ADIABATIC LIMIT OF THE ETA INVARIANT OVER COFINITE QUOTIENT OF $\mathrm{PSL}(2,\mathbb{R})$

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Abstract. We study the adiabatic limit of the eta invariant of the Dirac operator over cofinite quotient of $\mathrm{PSL}(2,\mathbb{R})$, which is a *noncompact* manifold with a *nonexact* fibred-cusp metric near the ends.

1. Introduction

The eta invariant of a Dirac operator was introduced in the seminal paper [2] as the boundary correction term in the Atiyah-Patodi-Singer index formula. Since the paper of Atiyah *et al.* [2], the eta invariant has found a place in many branches of mathematics. One particular aspect of the eta invariant which has found fruitful applications is the study of the *adiabatic limit* of the eta invariant, in which the eta invariant on the total space of a fibration is investigated when the fiber is collapsed. This was initiated by Witten [27] and later proved independently by Bismut and Freed [5] and Cheeger [8]. Expanding on the earlier work of [5, 8, 27], Bismut and Cheeger [7] and then Dai [9] studied the adiabatic limit for a general fibration of compact manifolds. We also refer to the recent work of Moroianu [21], who analyzed the adiabatic limit for general families of first order elliptic operators from the view point of the calculus of adiabatic pseudodifferential operators.

In this paper we study the spectral properties of the Dirac operator and compute the adiabatic limit of its eta invariant on a certain three-dimensional noncompact manifold X, which is given by a cofinite quotient of $\mathrm{PSL}(2,\mathbb{R})$. Then X is a circle bundle that fibers over a *non-compact* Riemann surface Σ with cusps,

$$\begin{CD} S^1 @>>> X \\ @. @VVV \\ @. \Sigma_{g,\kappa}\,, \end{CD} \tag{1.1}$$

where g, κ denote the genus and the number of cusps, respectively, of the base Riemann surface which has finite volume. There has been much interest in more general spectral problems for the case when the fibers are circles over compact bases [1, 4, 10, 13, 22, 23, 25, 29]. However, in all these papers the base manifold is compact. In this paper we consider the spectral properties and the eta invariant in the case when the base is not compact.

Replacing the circle S^1 in (1.1) with a circle of radius r, then choosing a spin structure, we denote the corresponding Dirac operator by D_r. The first purpose of this paper is to study the spectral properties of D_r defined by non-exact fibred cusp metrics near the ends. In particular, we analyze the dependence of the continuous spectrum of D_r on the choice of

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spin structure. Postponing terminology concerning trivial spin structures to Section 2, we can now state our first result:

Theorem 1.1. *With κ^{t} denoting the number of ends with trivial spin structures, the continuous spectrum of the Dirac operator D_r consists of κ^{t}-copies of countably many families given by*

$$\Big(-\infty, -\frac{r}{2} - |m|\big(1+r^{-2}\big)^{1/2}\Big] \bigcup \Big[-\frac{r}{2} + |m|\big(1+r^{-2}\big)^{1/2}, +\infty\Big)$$

indexed by odd integers $m \in 1+2\mathbb{Z}$ if the spin structure along the S^1-fiber is trivial, or by even integers $m \in 2\mathbb{Z}$ otherwise.

This theorem can be regarded as a generalization of the result of Bär in [3] to the fibred cusp case where the continuous spectrum of D_r depends on the spin structures of the S^1-fibers and of the S^1 cross sections of the base manifold $\Sigma_{g,\kappa}$ near the ends. Another novelty of this theorem is that the Riemannian metric over the cusps are *not* exact fibred cusp metrics, which have been extensively studied in, for example, [16, 17, 26]. It is because of the non-exact fibred cusp metrics that the continuous spectrum of D_r is quite complicated.

The second main result of this paper is the adiabatic limit of the eta invariant of D_r as the fiber is collapsed (that is, $r \to 0$). As we observed in Theorem 1.1, the Dirac operator D_r has continuous spectrum as well as discrete spectrum; moreover, the corresponding odd heat kernel of D_r^2 is not trace class. Therefore, the standard definition of the eta function using the eigenvalues or the trace of the odd heat kernel are not valid in our situation. This requires us to define a "regularized" eta invariant, which is reminiscent of the b-eta invariant of Melrose [20] and similar to a regularized eta invariant used by Park [24] to analyze eta invariants on hyperbolic manifolds with cusps. With $\eta(D_r, s)$ denoting the eta function of D_r defined through a regularized trace similar to Melrose's b-trace [20] (see Definition in (4.5)), the following is our main result:

Theorem 1.2. *We assume that the spin structure along the S^1-fiber is trivial.*

1. *For sufficiently small $r > 0$, $\eta(D_r, s)$ defined for $\Re(s) \gg 0$ has a meromorphic extension over $\mathbb{C}$ and may have a double pole at $s = 1$ and simple poles on $-\mathbb{N} \cup \{0, 2\}$. Moreover, for a totally nontrivial spin structure, $\eta(D_r, s)$ may have only simple poles at $-\mathbb{N} \cup \{0, 1, 2\}$.*
2. *For the eta invariant,*

$$\eta(D_r) := \mathrm{Reg}_{s=0}\, \eta(D_r, s)$$

where $\mathrm{Reg}_{s=0}$ means to take the regular value at $s = 0$, the following equality holds:

$$\lim_{r\to 0} \eta(D_r) = -\frac{1}{12\pi}\mathrm{Vol}(\Sigma_{g,\kappa}) = \frac{1}{6}\,(2 - 2g - \kappa) \tag{1.2}$$

where $\mathrm{Vol}(\Sigma_{g,\kappa})$ is given w.r.t. the Poincaré metric.

For the compact case, and for the trivial spin structure, a result corresponding to the formula (1.2) in Theorem 1.2 was proved by Seade and Steer [25], who also obtained the original value of the eta invariant by applying the APS index formula for a manifold with smooth boundary. In our noncompact case, obtaining the original value of the eta invariant will require an index formula for manifolds whose boundaries are manifolds with non-exact fibred cusp ends. This problem will be considered elsewhere.

The paper is organized as follows. In Section 2 we develop the required background material, including a discussion of spin structures and the Dirac operator D_r. In Section 3 we

analyze the Dirac operator in terms of the fibred cusp calculus of Mazzeo–Melrose [19] and we prove Theorem 1.1. In Section 4 we define the regularized eta invariant and in Sections 5, 6, and 7 we analyze the geometric and spectral sides of the Selberg trace formula in our context, which will be used to prove Theorem 1.2.

2. Dirac operator and Spin structure

In this section, we define the Dirac operator over the cofinite quotient of $\mathrm{PSL}(2,\mathbb{R})$ by a discrete subgroup. Equivalently we consider the Lie group $G = \mathrm{SL}(2,\mathbb{R})$ and a discrete subgroup $\Gamma \subset G$ containing $\{\pm 1\}$, so that the quotient $\Gamma\backslash G$ is the same as the quotient $(\Gamma/\{\pm 1\})\backslash\mathrm{PSL}(2,\mathbb{R})$.

For $r \in (0,\infty)$ we define a family of metrics g_r over G such that the left translations of $E := r^{-1}C, A, H$ are orthonormal w.r.t. g_r where C, A, H is a basis of $\mathfrak{g} = \mathrm{sl}(2,\mathbb{R})$ given by

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}$$

Recall that the corresponding Levi-Civita connection ∇^r is determined by the Koszul formula

$$\begin{aligned} 2g_r(\nabla^r_X Y, Z) = X g_r(Y,Z) + Y g_r(Z,X) - Z g_r(X,Y) \\ + g_r([X,Y],Z) - g_r([X,Z],Y) - g_r([Y,Z],X) \end{aligned}$$

where X, Y, Z denote vector fields over G.

Since G is topologically the same as $S^1 \times \mathcal{H}$ where $\mathcal{H}$ is the Poincaré upper half plane, there are two spin structures on G. We choose the one determined by the left invariant trivialization. Denoting the lifted connection to spinor bundle by the same notation ∇^r, we define the Dirac operator by

$$\widehat{D}_r := E.\nabla^r_E + A.\nabla^r_A + H.\nabla^r_H$$

where $X.$ denotes the Clifford action by X. By a straightforward computation as in [13], [25], we obtain

$$\widehat{D}_r\psi = \frac{1}{2}\big(\frac{2+r^2}{r^2} - 2\big)C.A.H.\psi$$

for a basic spinor ψ.

We twist $\widehat{D}_r$ by multiplying the volume element $\omega := E.A.H.$ to define $\widetilde{D}_r$, that is,

$$\widetilde{D}_r := E.A.H.\widehat{D}_r,$$

which has the following simplified form,

$$\widetilde{D}_r\psi = \Big(\frac{2-r^2}{2r}\Big)\psi$$

for a basic spinor ψ. The Clifford algebra generated by E, A, H has the *Pauli matrix representation* given by

$$E \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Then we have

$$\omega E \mapsto -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \omega A \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \omega H \mapsto i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From these, for any spinor $\alpha\psi_1 + \beta\psi_2$ with basic spinors ψ_1, ψ_2 and smooth functions α, β on G, we have the following representation of $\widetilde{D}_r$,

$$\widetilde{D}_r \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \Big(\frac{2-r^2}{2r}\Big) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} -iE & A+iH \\ -A+iH & iE \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{2.2}$$

Now we let

$$Z := -iC, \qquad 2X_+ := A - iH, \qquad 2X_- := A + iH$$

(Our convention is slightly different from the one in [25]). Then they satisfy

$$[Z, X_+] = 2X_+, \qquad [Z, X_-] = -2X_-, \qquad [X_+, X_-] = Z,$$

and we have

$$\widetilde{D}_r = \Big(\frac{2-r^2}{2r}\Big) + \begin{pmatrix} r^{-1}Z & 2X_- \\ -2X_+ & -r^{-1}Z \end{pmatrix} \quad \text{acting on} \quad C^\infty(G) \oplus C^\infty(G).$$

It is also easy to check that

$$\widetilde{D}_r^2 = \begin{pmatrix} -(A^2 + H^2 + r^{-2}C^2) & 0 \\ 0 & -(A^2 + H^2 + r^{-2}C^2) \end{pmatrix} + \text{ lower order terms}, \tag{2.3}$$

hence the Dirac Laplacian $\widetilde{D}_r^2$ is a generalized Laplacian whose principal symbol is given by the metric g_r.

To define the Dirac operator over $X = \Gamma\backslash G$, let us discuss on the spin structures on $X = \Gamma\backslash G$, which will play a crucial role throughout this paper. First recall that there are $|H^1(X, \mathbb{Z}_2)|$-number of spin structures over X since every 3-dimensional manifold is spin. This can be understood from the following diagram,

$$\begin{array}{ccc} \widetilde{\mathcal{S}} & \longrightarrow & \mathcal{S} \\ \Big\downarrow & & \Big\downarrow \\ \widetilde{X} & \xrightarrow{\pi} & X \end{array}$$

where $\widetilde{\mathcal{S}}, \mathcal{S}$ are the Spin bundles over the universal covering manifold $\widetilde{X}$ and X respectively. Since $\widetilde{\mathcal{S}} \cong \widetilde{X} \times \mathrm{Spin}(3)$ and $\widetilde{\mathcal{S}} \cong \pi^*\mathcal{S}$, the possible Spin bundle $\mathcal{S}$ is given by the $\mathbb{Z}_2$-representation ρ of $\pi_1(X)$ as follows:

$$\mathcal{S}_\rho = \widetilde{X} \times_\rho \mathrm{Spin}(3) \tag{2.4}$$

with the obvious $\mathbb{Z}_2$-action to Spin(3). Therefore, each $\mathbb{Z}_2$-representation of $\pi_1(X)$ provides us with inequivalent spin structure on X. Recall

$$\pi_1(X) = \Big\{\, x_i, y_i, h_j, k \;\Big|\; 1 \le i \le g,\, 1 \le j \le \kappa, \\ \prod_{i=1}^{g} [x_i, y_i] \prod_{j=1}^{\kappa} h_i = 1,\; [x_i, k] = [y_i, k] = [h_j, k] = 1 \,\Big\},$$

where g, κ denote the number of genus and cusps of the base Riemann surface $\Sigma_{g,\kappa}$ of the fibration (1.1).

Among spin structures, there are spin structures which are determined by those $\mathbb{Z}_2$-representations ρ of $\pi_1(X)$ with $\rho(h_j) = -1$ for some j. Such a spin structure over Riemann surface $\Sigma_{g,\kappa}$ is called *nontrivial along the cusp* corresponding to j in [3]. Following [3], we

call such a spin structure *nontrivial along the cusp* if $\rho(h_j) = -1$ for the corresponding j, and *totally nontrivial spin structure* if it is determined by a $\mathbb{Z}_2$-representation ρ with

$$\rho(h_j) = -1 \text{ for all } j = 1, \dots, \kappa. \tag{2.5}$$

From the relation of the generators of $\pi_1(X)$, there is the following obstruction for this,

$$\prod_{j=1}^{\kappa} \rho(h_j) = 1.$$

Hence, in this case, the number of cusps κ should be even. We distinguish two classes of spin structures according to (non)triviality of spin structure along the fiber $S^1/\{\pm 1\}$. We call the spin structure *trivial along the fiber* if the spin structure is trivial along the fiber $S^1/\{\pm 1\}$ (or equivalently, if the representation ρ maps the generator k to 1), and *nontrivial along the fiber* otherwise. Note that if the spin structure is trivial along the $S^1/\{\pm 1\}$-fiber, this spin structure does not extend to a spin structure over the disc bundle over $\Sigma_{g,\kappa}$. From the above discussion we have

Proposition 2.1. *There are $2^{2g+\kappa}$ spin structures over $X = \Gamma\backslash G$. There exist totally nontrivial spin structures over X if and only if κ is even.*

For the trivial representation of $\pi_1(X)$, the resulting Spin bundle denoted by $\mathcal{S}_1$ is topologically trivial, determined by the left invariant trivialization over $X = \Gamma\backslash G$. The associated spinor bundle $\Sigma_1 = \mathcal{S}_1 \times_{\mathrm{Spin}(3)} \Sigma(3)$ (where $\Sigma(3) \cong \mathbb{C}^2$ is the spinor representation of $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$) is therefore also trivial and has the following relation with other Σ_ρ,

$$\Sigma_\rho = \Sigma_1 \otimes \mathbb{C}_\rho$$

where $\mathbb{C}_\rho \to X$ is the flat line bundle associated to ρ.

From the definition of $\widetilde{D}_r$ over G and the equality (2.4), we can see that the induced Dirac operator from $\widetilde{D}_r$ pushed down to X has the following form,

$$D_r = \Big(\frac{2-r^2}{2r}\Big) + \begin{pmatrix} r^{-1}Z & 2X_- \\ -2X_+ & -r^{-1}Z \end{pmatrix} \quad \text{acting on} \quad C_0^\infty(\Gamma\backslash G, \chi) \tag{2.6}$$

where $C_0^\infty(\Gamma\backslash G, \chi)$ ($\chi = \rho \oplus \rho$) consists of the smooth functions with co-compact supports such that $f(\gamma x) = \chi(\gamma) f(x)$ for $\gamma \in \pi_1(X)$, $x \in G$. We also denote the L^2-completion of D_r (w.r.t. certain metric metric explained in (4.1)) by D_r, that is,

$$D_r \,:\, L^2(\Gamma\backslash G, \chi) \longrightarrow L^2(\Gamma\backslash G, \chi). \tag{2.7}$$

3. Analysis of fibred cusp operators

In this section we show that the metrics g_r are of *conformal fibred cusp* type. Consequently, we show that the Dirac operators D_r belong to the class of weighted fibred cusp operators introduced by Mazzeo and Melrose [19], and we prove Theorem 1.1.

First we introduce some subgroups of $G = \mathrm{SL}(2, \mathbb{R})$,

$$N_0 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}, A_0 = \left\{ \begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} \,\middle|\, u \in \mathbb{R} \right\}, K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}. \tag{3.1}$$

Then the standard parabolic subgroup P_0 is given by $N_0 A_0 Z$ where $Z = \{\pm 1\} \subset K$ and any *parabolic subgroup* P of G is conjugate to P_0 by an element k_P in K. A parabolic subgroup P has a decomposition $P = N_P A Z$ where N_P is the derived group of P and A is any conjugate

of A_0 in P, to be called a *Cartan subgroup*. It is clear that A_0 is the unique Cartan subgroup P_0 with Lie algebra orthogonal to that of K. Therefore, P has a unique Cartan subgroup with the same property. From now on, we assume that the pair (P, A) satisfies this condition. For such a pair (P, A) with $N = N_P$, we have the *Iwasawa decomposition* $G = NAK$.

For a given $\Gamma \subset G$, a parabolic subgroup P of G is called Γ-*cuspidal* if $N = N_P$ contains a nontrivial element of Γ. As one knows, the finitely many ends of $X \cong \Gamma\backslash G$ are parametrized by Γ-conjugacy classes $\{P\}_\Gamma = \{\gamma P\gamma^{-1} \mid \gamma \in \Gamma/\Gamma_P\}$ where $\Gamma_P := \Gamma\cap P$. Let P be a Γ-cuspidal parabolic subgroup of G corresponding to one end of $X = \Gamma\backslash G$. This subgroup determines a cusp c_P, an incomplete manifold which is identified with a neighborhood of the cuspidal end of the quotient $\Gamma_P\backslash G$.

Assume first that $P = P_0$ is the standard parabolic subgroup of G. The manifold $\Gamma_P\backslash G$ has two commuting free S^1 actions: the action of K to the right and that of $\Gamma_{N_0}\backslash N_0$ to the left where $\Gamma_{N_0} := \Gamma \cap N_0$. The first S^1 action exists in fact globally on $X = \Gamma\backslash G$, while the second one exists only on the cusp. Let $\gamma_l := \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$ be the generator of Γ_{N_0}. Then $\Gamma_P\backslash G$ is identified with $\mathbb{R}/l\mathbb{Z} \times \mathbb{R} \times \mathbb{R}/2\pi\mathbb{R}$ by the map

$$(x, u, \theta) \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.2}$$

By projection on the last two terms, we view this as the total space of a fibration with fiber S^1. Note that this fibration makes sense only near the end and is not the fibration in (1.1) where the roles of two S^1's are reversed.

As seen above, the spinor bundle corresponding to the representation ρ is the spinor bundle for the trivial representation, twisted by the flat line bundle $\mathbb{C}_\rho$ defined by ρ. The Dirac operator on G has been computed in (2.6) with respect to the orthonormal vector fields $r^{-1}C$, A, H defined in (2.1) and the representation χ. The same expression holds on the spinor bundle on the cusp $\Gamma_P\backslash G$, where the vector fields $r^{-1}C$, A, H now act on Σ_ρ. There is no ambiguity about the action of these vector fields since the twisting bundle $\mathbb{C}_\rho$ is flat.

Introduce the function $\nu := e^{-u}$ on the cusp and glue the "boundary at infinity" $\mathbb{R}/l\mathbb{Z}\times\{\nu = 0\} \times \mathbb{R}/2\pi\mathbb{R}$ to the cusp, thus getting a manifold with boundary $\overline{\Gamma_P\backslash G}$. The S^1-fibration structures extend to the boundary. We will show that for each fixed r, the metric g_r on X is conformal to a fibred cusp metric (with respect to the fibration of the boundary induced from the $\Gamma_{N_0}\backslash N_0$ action). In the coordinates $(x, \nu = e^{-u}, \theta)$ of $\Gamma_P\backslash G$, the coefficients of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given by the inverse of the map (3.2):

$$x = \frac{ac + bd}{c^2 + d^2}, \qquad \nu = c^2 + d^2, \qquad \theta = -\arctan\left(\frac{c}{d}\right).$$

We compute then

$$\begin{aligned} E &= \ \ r^{-1}\partial_\theta, \\ A &= -\cos 2\theta\partial_\theta + 2\nu^{-1}\cos 2\theta\partial_x - 2\nu\sin 2\theta\partial_\nu, \\ H &= \ \ \sin 2\theta\partial_\theta - 2\nu^{-1}\sin 2\theta\partial_x - 2\nu\cos 2\theta\partial_\nu. \end{aligned} \tag{3.3}$$

These equalities also can be found at p.52 of [12] or p.115 of [15]. It follows that in the coordinates (x, ν, θ) the metric g_r is given by

$$\frac{1}{4\nu^2}d\nu^2 + \frac{\nu^2}{4}dx^2 + r^2(d\theta + \frac{\nu}{2}dx)^2,$$

thus

$$g_\Phi := \frac{4}{\nu^2}g_r = \left(\frac{d\nu}{\nu^2}\right)^2 + r^2\left(\frac{2d\theta}{\nu} + dx\right)^2 + dx^2.$$

This is what is called a fibred cusp metric, or a smooth metric on the fibred cusp tangent bundle. To define this, consider the subalgebra ${}^\Phi\mathcal{V}$ of the Lie algebra $\mathcal{V}$ of smooth vector fields on the manifold with boundary $\overline{\Gamma_P\backslash G}$, spanned over $C^\infty(\overline{\Gamma_P\backslash G})$ by the vector fields

$$V_\nu := \nu^2\partial_\nu, \qquad V_\theta := \nu\partial_\theta, \qquad V_x := \partial_x.$$

This sub-algebra is by definition a free $C^\infty(\overline{\Gamma_P\backslash G})$-module so it is the space of sections of a smooth vector bundle over $C^\infty(\overline{\Gamma_P\backslash G})$; this vector bundle, called ${}^\Phi T\Gamma_P\backslash G$, comes equipped with a bundle morphism to the usual tangent bundle $T\overline{\Gamma_P\backslash G}$, induced from the inclusion of the spaces of sections ${}^\Phi\mathcal{V} \hookrightarrow \mathcal{V}$, which is an isomorphism over $\Gamma_P\backslash G$.

Since ${}^\Phi\mathcal{V}$ is a Lie algebra and the metric g_Φ defined above is non-degenerate and smooth on fibred cusp vector fields, it follows immediately from the Cartan formula that the Levi-Civita connection on $\Gamma_P\backslash G$ with respect to the metric g_Φ extends to the boundary in the sense that for every $V_i, V_j, V_k \in {}^\Phi\mathcal{V}$, we have

$$\langle\nabla_{V_i}V_j, V_k\rangle \in C^\infty(\overline{\Gamma_P\backslash G}).$$

The spinor bundle Σ_ρ extends over the boundary, such that the Clifford multiplication by V_i is a smooth map. Take now the orthonormal frame

$$V_1 := V_\nu, \qquad V_2 := V_x - V_\theta/2, \qquad V_3 := \frac{1}{2r}V_\theta.$$

Its relationship to the global frame (E, A, H) is deduced from (3.3):

$$\begin{aligned} V_1 &= -\frac{\nu}{2}(\sin 2\theta A + \cos 2\theta H) \\ V_2 &= \ \ \frac{\nu}{2}(\cos 2\theta A - \sin 2\theta H) \\ V_3 &= \ \ \frac{\nu}{2}E. \end{aligned} \tag{3.4}$$

Denote by V a local lift to the spinor bundle of the orthonormal frame (V_1, V_3, V_2). Let $\sigma : \overline{\Gamma_P\backslash G} \to \Sigma(3)$ be a smooth map into the 3-spinor representation space. It follows from the local formula

$$\begin{aligned} D_\Phi[V,\sigma] &= \sum_{i=1}^{3} c(V_i)\left([V, V_i(\sigma)] + \frac{1}{2}\sum_{j<k} c(V_j)c(V_k)\langle\nabla_{V_i}V_j, V_k\rangle\right) \\ &= \left(c(V_1)(\nu^2\partial_\nu - \frac{\nu}{2}) + c(V_2)(\partial_x - \frac{\nu\partial_\theta}{2}) + c(V_3)\partial_\theta\frac{\nu}{2r} - r\frac{\nu}{4}\right)[V,\sigma] \end{aligned} \tag{3.5}$$

that the Dirac operator with respect to g_Φ (defined first on compactly supported spinors over $\Gamma_P\backslash G$) extends to smooth spinors up to the boundary. Such an operator, a combination of

fibred cusp vector fields and of smooth bundle endomorphisms down to the boundary $\{\nu = 0\}$, is called a fibred cusp differential operator. Thus

$$D_\Phi \in \mathrm{Diff}^1_\Phi(\overline{\Gamma_P\backslash G}, \Sigma_\rho).$$

The Dirac operator changes very nicely with respect to conformal changes of the metric. We simply have

$$D_r = 2\nu^{-2} \circ D_\Phi \circ \nu$$

so for $r > 0$, the Dirac operator D_r is a differential operator in $\nu^{-1}\mathrm{Diff}^1_\Phi(\overline{\Gamma_P\backslash G}, \Sigma_\rho)$.

The *normal operator* $\mathcal{N}(D_\Phi)(\theta, \xi, \tau)$ of D_Φ (see [19]) is obtained by replacing formally

$$V_\nu \mapsto i\xi, \qquad\qquad V_\theta \mapsto i\tau$$

and then restricting to $\nu = 0$. The result is a family of differential operators on the fibers of the boundary fibration (the x-circles) with coefficients in the spinor bundle, with parameters $\theta \in S^1$, $(\xi, \tau) \in \mathbb{R}^2$:

$$\mathcal{N}(D_\Phi)(\theta, \xi, \tau) = c(V_1)i\xi + c(V_2)\left(\partial_x - \frac{i\tau}{2}\right) + c(V_3)\frac{i\tau}{2r}.$$

Definition 3.1. The operator D_Φ is called *fully elliptic* if $\mathcal{N}(D_\Phi)(\theta, \xi, \tau)$ is invertible for all $(\theta, \xi, \tau) \in S^1 \times \mathbb{R}^2$.

If D_Φ is fully elliptic, then by the results of [19] it has a parametrix inside the calculus of fibered cusp pseudodifferential operators $\Psi^{-1}_\Phi(X)$ modulo compact operators.

Proof of Theorem 1.1. According to the decomposition principle (see e.g. [3, Proposition 1]), the essential spectrum of D_r can be computed outside a compact subset of X, thus it is a superposition of the essential spectra of any self-adjoint extension of D_r over each cuspidal end c_P defined by $\nu_P < \epsilon_P$. We must make sure that such an extension exists (the Dirac operator on a manifold with boundary may not admit self-adjoint extensions, e.g. on $\mathbb{R}^+$). We may take for instance the Atiyah-Patodi-Singer boundary condition at the torus boundary $\{\nu_P = \epsilon_P\}$. Special care is needed for the nullspace of the Dirac operator along the torus, we allow in the domain only harmonic spinors of the form $(u, c(V_3)u)$ where u is in the i-eigenspace of $c(V_1)$.

Since any Γ-parabolic subgroup P is conjugated by an element in the maximal compact subgroup K to the standard parabolic subgroup P_0, we see that the cusp corresponding to P is isometric to the "canonical" cusp $P_0\backslash G$. Thus we can assume that we work with the canonical parabolic subgroup P_0.

We have seen above that D_r belongs to $\nu^{-1}\mathrm{Diff}^1_\Phi(\overline{\Gamma_P\backslash G}, \Sigma_\rho)$ near the cuspidal end.

Lemma 3.2. *The fibered cusp operator D_Φ is fully elliptic on the cusp c_P if and only if the spin structure is non-trivial along c_P.*

Proof. We have computed above the normal operator $\mathcal{N} := \mathcal{N}(D_\Phi)(\theta, \xi, \tau)$. Clearly, $\mathcal{N}$ is an elliptic self-adjoint operator on the circle in the variable x. Therefore $\mathcal{N}$ is invertible if and only if $\mathcal{N}^2$ is. Now by the anti-commutation of the Clifford variables,

$$\mathcal{N}^2 = \xi^2 + \frac{\tau^2}{4r^2} + (i\partial_x - \frac{\tau}{2})^2.$$

This family of operators is independent of θ; it is strictly positive (hence invertible) for $(\xi, \tau) \neq 0 \in \mathbb{R}^2$. For $\xi = \tau = 0$, $\mathcal{N} = -\partial_x^2$, so $\ker(\mathcal{N})$ is made of those spinors which are constant in x in the trivialization V of the spinor bundle. For fixed θ, such spinors exist

globally on the x-circle if and only if the local lift V satisfies $V_{x=l} = V_{x=0}$. Now the frame (V_1, V_2, V_3) is obtained from (E, A, H) by the transformation (3.4) which is constant in x; thus the lift V exists globally around the cusp if and only if the lift of (E, A, H) exists globally around the cusp, which is by definition equivalent to the triviality of the spin structure around the cusp c_P. □

If D_Φ is fully elliptic, it follows from the above lemma and from the general theory of fibred cups operators that $D_r = 2\nu^{-2} D_\Phi \nu$ has a parametrix $Q \in \nu\Psi_\Phi^{-1}$ over the cusp c_P, modulo compact operators. But Q itself is compact due to the decaying weight ν; hence the self-adjoint operator $\mathcal{D}_r$ has pure-point spectrum over the cusp.

Conversely, assume that the spin structure is trivial along the cusp. The operator $D_r = 2\nu^{-2} D_\Phi \nu$ computed in (3.5) has constant coefficients in x, thus it preserves the orthogonal decomposition into zero-modes and high-energy modes

$$L^2(\Gamma_P \backslash G \cap \{\nu_P < \epsilon_P\}, \Sigma_\rho) =: \mathcal{H}_0 \oplus \mathcal{H}'$$

where $\mathcal{H}_0$ is the space of spinors constant in x in the trivialization V (we have seen above that V exists globally around the cusp if the spin structure is trivial along c_P) and $\mathcal{H}'$ its orthogonal complement. Over $\mathcal{H}'$, by the same argument as above, there exists a compact parametrix of D_r inside the fibered cusp calculus. Thus the essential spectrum of D_r over the cusp c_P only arises from the zero-modes, i.e., it is the essential spectrum of the operator

$$2\nu^{-1}\left(c(V_1)(\nu\partial_\nu - \frac{1}{2}) - c(V_2)\frac{\partial_\theta}{2} + c(V_3)\frac{\partial_\theta}{2r} - \frac{r}{4}\right)\nu$$

acting in $L^2([0,\epsilon) \times S^1, \Sigma_\rho, d\nu d\theta)$ with any boundary condition at ϵ which makes it self-adjoint. We conjugate this operator through the Hilbert space isometry

$$L^2(d\nu d\theta) \to L^2\left(\frac{d\nu}{\nu} d\theta\right) \qquad\qquad \phi \mapsto \nu^{\frac{1}{2}}\phi.$$

We get the operator

$$A_r = 2c(V_1)\nu\partial_\nu + \left(\frac{c(V_3)}{r} - c(V_2)\right)\partial_\theta - \frac{r}{2}.$$

This can be again decomposed according to the frequencies in the θ variable. Note that although the local lift V may not exist globally, the ambiguity is locally constant so the operator $i\partial_\theta$ is well-defined; moreover, it clearly commutes with A_r.

From (3.4), the frame (V_1, V_2, V_3) is obtained (after rescaling) from the frame (E, A, H) by a complete rotation around the E axis in time π. Such a rotation is a generator of $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$. Hence V exists globally around the θ circle if and only if the lift of (E, A, H) does not, i.e., if the spin structure is non-trivial along the fiber $S^1/\{\pm 1\}$. Otherwise, if the spin structure is trivial along the fiber $S^1/\{\pm 1\}$, then after time π the lift V changes sign.

A spinor $[V, \sigma]$ is in the m-eigenspace of $i\partial_\theta$ if and only if

$$\sigma(t + \theta) = e^{-im\theta}\sigma(t). \tag{3.6}$$

The resulting spinor should be π-periodic (since we work on $\mathrm{PSL}(2, \mathbb{R})$, we assumed that $-1 \in \Gamma$). We distinguish two cases:

- The spin structure is nontrivial along the $S^1/\{\pm 1\}$ fiber. Then $V(\pi) = V(0)$ so we want $\sigma(\pi) = \sigma(0)$. The eigenspinor equation (3.6) gives $m \in 2\mathbb{Z}$;

- The spin structure is trivial along the $S^1/\{\pm 1\}$ fiber. Then $V(\pi) = -V(0)$ so we want $\sigma(\pi) = -\sigma(0)$. The eigenspinor equation (3.6) gives $m \in 1 + 2\mathbb{Z}$.

In both cases, the m-eigenspaces are 2-dimensional representation spaces for $c(V_j)$, $j = 1, 2, 3$.

Denote by $A_{r,m}$ the action of A_r on the m-eigenspace of $i\partial_\theta$. We get a b-operator $A_{r,m}$ (in the sense of Melrose) in $L^2([0, \epsilon), \mathbb{C}^2, \nu^{-1}d\nu)$

$$A_{r,m} = 2c(V_1)\nu\partial_\nu - c\left(\frac{V_3}{r} - V_2\right) im - \frac{r}{2}.$$

The b-normal operator of $A_{r,m}$ is obtained by replacing $\nu\partial_\nu$ with is where s is a complex parameter. One knows from the general theory of b-operators [20] that the following statements are equivalent:

- λ does not belong to the essential spectrum of $A_{r,m}$, i.e., $A_{r,m} - \lambda$ is Fredholm;
- $\mathcal{N}(A_{r,m})(s) - \lambda$ is invertible for all $s \in \mathbb{R}$.

We use now the representation

$$c(V_1) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad c\left(\frac{V_3}{r} - V_2\right) = \begin{bmatrix} 0 & 1 + r^{-2} \\ -1 & 0 \end{bmatrix}$$

so that

$$\mathcal{N}(A_{r,k})(s) = \begin{bmatrix} -2s & -im(1 + r^{-2}) \\ im & 2s \end{bmatrix} - \frac{r}{2}.$$

An easy computation shows that $\mathcal{N}(A_{r,k})(s) - \lambda$ is invertible for all $s \in \mathbb{R}$ exactly for

$$\lambda \in \left(-\frac{r}{2} - |m|(1 + r^2)^{\frac{1}{2}} r^{-1},\ -\frac{r}{2} + |m|(1 + r^2)^{\frac{1}{2}} r^{-1}\right).$$

Thus the essential spectrum of D_r is the superposition of the complements of these intervals for each k and for each cusp c_p with trivial spin structure.

In the sequel, we will assume that the spin structure is trivial along the $S^1/\{\pm 1\}$ fiber, so the essential spectrum does not touch 0 for small $r > 0$. In this case an alternate proof of Theorem 1.1 will follow from a computation using harmonic analysis over G (see Section 5).

4. Regularized trace and Geometric side

In this section, we study the relation of certain regularized trace of the odd heat operator of D_r with the geometric side of the Selberg trace formula.

To use the harmonic analysis over G, we need to fix the Haar measures over G and its subgroups. First the parametrizations in (3.1) for A_0, N_0 carry the Lebesgue measure du, dn from $\mathbb{R}$ to A_0, N_0. Now we fix Haar measures on K by $\mathrm{vol}(K/Z) = 1$ and on G by

$$\int_G f(g)\, dg = \int_{N_0} \int_{A_0} \int_K f(na_u k) e^{-u}\, dk\, du\, dn \tag{4.1}$$

for $f_0 \in C_0(G)$ and $a_u = \mathrm{diag}(e^{\frac{u}{2}}, e^{-\frac{u}{2}})$. For $a_{P,u} := k_P^{-1} a_u k_P \in A = k_P^{-1} A_0 k_P$, we put

$$H_P(g) = u \qquad \text{for} \quad g \in N a_{P,u} K.$$

The Iwasawa decomposition $\mathcal{H} \cong G/K \cong NA$ provides a parametrization of the geodesics $nA \cdot i \subset \mathcal{H}$ to the infinity. The parameter value is given by the function H_P whose potential curves are N-orbits (horocycles) on $\mathcal{H}$. However, this parametrization is not adapted to Γ. To rectify this, we replace k_P by $g_P = a_{u_P} k_P$ where $e^{-u_P} = \mathrm{vol}(\Gamma_N \backslash N)$ where $\Gamma_N := \Gamma \cap N$. For the new parameter

$$H_P(g) + u_P = H_{P_0}(g_P g),$$

then the value 0 of this new parameter corresponds to the horocycle whose projection on $\Gamma\backslash\mathcal{H}$ has length 1.

For $\phi \in \mathsf{H} := L^2(Z\backslash K) = \langle e^{im\theta} \,|\, m \in 2\mathbb{Z}\rangle$ and $s \in \mathbb{C}$, we extend ϕ to G by

$$\phi_s(na_uk) = e^{su}\phi(k) \qquad \text{for} \quad n \in N_0, k \in K.$$

These functions constitute the Hilbert space $\mathsf{H}_s \cong \mathsf{H}$ in which the representation π_s induced from the parabolic subgroup $P_0 = N_0A_0Z$ acts as

$$(\pi_s(g)\phi_s)(x) = \phi_s(xg).$$

From now on, we assume that $\mathfrak{P} = \{P_1, \ldots, P_\kappa\}$ is a set of representatives for Γ-conjugacy classes of the cuspidal parabolic subgroups and that the spin structure over c_{P_i} for $1 \le i \le \kappa^{\mathrm{t}}$ is trivial. We also assume that the representation ρ maps the generator $k \in \pi_1(X)$ to the identity 1, thus we consider only spin structures which are trivial along the $S^1/\{\pm 1\}$-fiber. For the representation space $V \cong \mathbb{C}^2$ of $\chi = \rho \oplus \rho$, we let V^P be the invariant subspace of V under the action $\chi|_{\Gamma_P}$. Then

$$V^{P_i} = \begin{cases} V & \text{if} \quad 1 \le i \le \kappa^{\mathrm{t}} \\ \{0\} & \text{if} \quad \kappa^{\mathrm{t}} + 1 \le i \le \kappa \end{cases} .$$

For a cuspidal parabolic subgroup P, $s \in \mathbb{C}$ with $\Re(s) > 1$ and $\phi \in \mathsf{H} \otimes V^P$, the Eisenstein series $E(P, \phi, s)$ is defined by

$$E(P, \phi, s)(g) := \sum_{\gamma \in \Gamma/\Gamma_P} \chi(\gamma)\phi_s(g_P\gamma^{-1}g).$$

Note that there is no Eisenstein series attached to P_i if $\kappa^{\mathrm{t}} + 1 \le i \le \kappa$. The Eisenstein series $E(P, \phi, s)$ converges absolutely and locally uniformly for $\Re(s) > 1$ and has the meromorphic extension over $\mathbb{C}$. In particular, $E(P, \phi, s)$ is an *automorphic form*, that is,

$$E(P, \phi, s)(\gamma g) = \chi(\gamma)E(P, \phi, s)(g) \qquad \text{for} \quad \gamma \in \Gamma, g \in G.$$

For $\phi \in H \otimes V^{\mathrm{cst}}$ with $V^{\mathrm{cst}} := \oplus_{P \in \mathfrak{P}} V^P$, we define

$$E(\phi, s) = \sum_{P \in \mathfrak{P}} E(P, \mathrm{pr}^P \phi, s) \tag{4.2}$$

where pr^P denotes the orthogonal projection onto V^P, and

$$E^{\mathrm{cst}}(\phi, s)(g) = \big(E^P(\phi, s)(g_P^{-1}g)\big)_{P \in \mathfrak{P}}.$$

Here, the *constant term* of $E^P(\phi, s)$ is defined by

$$E^P(\phi, s)(g) := \mathrm{vol}(\Gamma_N\backslash N)^{-1} \int_{\Gamma_N\backslash N} \mathrm{pr}^P\, E(\phi, s)(ng)\, dn$$

for $N = N_P$. Then we have

$$E^{\mathrm{cst}}(\phi, s) = \phi_s + \big(C(s)\phi\big)_{1-s}$$

where $C(s)$ is the *scattering operator* acting on $\mathsf{H} \otimes V^{\mathrm{cst}}$.

Now let us describe the spectral decomposition of $L^2(\Gamma\backslash G, \chi)$,

$$L^2(\Gamma\backslash G, \chi) = L^2(\Gamma\backslash G, \chi)_{\mathrm{cus}} \oplus L^2(\Gamma\backslash G, \chi)_{\mathrm{res}} \oplus L^2(\Gamma\backslash G, \chi)_{\mathrm{ct}}.$$

Here $L^2(\Gamma\backslash G, \chi)_{\mathrm{cus}}$ is the space of the cusp forms in $L^2(\Gamma\backslash G, \chi)$, and decomposes into a Hilbert direct sum of closed irreducible G-invariant subspaces with finite multiplicities. The

residual part $L^2(\Gamma\backslash G,\chi)_{\rm res}$ is the direct sum of the constants and of finitely many copies of the complementary series representation of G such that some Eisenstein series has a pole at $s\in(\frac12,1)$. These two spaces constitute the discrete part $L^2(\Gamma\backslash G,\chi)_{\rm dis}$. The continuous part $L^2(\Gamma\backslash G,\chi)_{\rm ct}$ is isometric to

$$\Big\{\,\Phi\in L^2(\frac12+i\mathbb{R},\frac{d\tau}{4\pi})\hat{\otimes}\mathsf{H}\otimes V^{\rm cst}\mid \Phi(\frac12-i\tau)=C(\frac12+i\tau)\Phi(\frac12+i\tau)\,\Big\}$$

by

$$\mathcal{I}^{\rm ct}\Phi=\frac{1}{4\pi}\int_{-\infty}^{\infty}E(\Phi,\frac12+i\tau)\,d\tau \tag{4.3}$$

where $E(\Phi,\frac12+i\tau)$ is defined as in (4.2) with $\phi=\Phi$ and $s=\frac12+i\tau$. For $f\in L^1(G)$, we define a representation on $L^2(\Gamma\backslash G,\chi)$ by

$$\pi(f):=\int_G f(g)\pi(g)\,dg$$

where π is the right translation action given by $\big(\pi(g)\phi\big)(x)=\phi(xg)$ for $\phi\in L^2(\Gamma\backslash G,\chi)$. We put $\pi^{\rm dis}(f)={\rm pr}^{\rm dis}\circ\pi(f)$, $\pi^{\rm ct}(f)={\rm pr}^{\rm ct}\circ\pi(f)$ where ${\rm pr}^{\rm dis},{\rm pr}^{\rm ct}$ denote the orthogonal projections onto $L^2(\Gamma\backslash G,\chi)_{\rm dis}, L^2(\Gamma\backslash G,\chi)_{\rm ct}$ respectively. In particular, $\pi^{\rm ct}(f)$ intertwines with $\pi^{\rm cst}_{\frac12+i\tau}(f):=\pi_{\frac12+i\tau}(f)\otimes{\rm Id}_{V^{\rm cst}}$ by $\mathcal{I}^{\rm ct}$ given in (4.3).

Since $\widetilde{D}_r$ is a left invariant differential operator, there is a function $\hat f_{t,r}\in C^\infty(G,M(2,\mathbb{C}))$ such that

$$\hat f_{t,r}(x^{-1}y)=\big(\widetilde{D}_r e^{-t\widetilde{D}_r^2}\big)(x,y)\qquad\text{for}\quad x,y\in G.$$

By the heat kernel estimates in [11], which also holds for the generalized Laplacian $\widetilde{D}_r^2$ with the form in (2.3), we have

$$||d_t^i\,d_x^j\,d_y^k\,\hat f_{t,r}(x^{-1}y)||\le Ct^{-\frac52-i-j-k}\,\exp\Big(-\frac{d_G^2(x,y)}{4t}\Big)$$

where C is a positive constant and d_G is the metric over G. (Note that we apply the method in [11] to a certain co-compact discrete subgroup Γ' in G to obtain the above estimate.) This estimate implies that $f_{t,r}:={\rm tr}(\hat f_{t,r})$ lies in the Harish-Chandra L^1-Schwartz space $\mathcal{C}^1(G)(\subset L^1(G))$ defined by

$$\begin{aligned}\mathcal{C}^1(G)=\Big\{f\in C^\infty(G)\;\Big|\;&|f(D_1k_{\theta_1}a_uk_{\theta_2}D_2)|\\ &\le Ce^{-|u|}(1+|u|+|\theta_1+\theta_2|)^{-n},\quad\forall n\in\mathbb{N},D_1,D_2\in\mathfrak{g}\Big\}\end{aligned}$$

where $f(D_1k_{\theta_1}a_uk_{\theta_2}D_2)$ denotes the convolution $D_1*\delta_{k_{\theta_1}}*\delta_{a_u}*\delta_{k_{\theta_2}}*D_2$ evaluated on f. Let us put

$$K(t,x,y):=\sum_{\gamma\in Z\backslash\Gamma}\hat f_{t,r}(x^{-1}\gamma y)\,\chi(\gamma)=\sum_{\gamma\in Z\backslash\Gamma}\widetilde{D}_re^{-t\widetilde{D}_r^2}(x,\gamma y)\,\chi(\gamma)\qquad\text{for } x,y\in G,$$

which is absolutely uniformly convergent on compact sets in G. For a Γ-cuspidal parabolic subgroup $P_j=P=NAZ$, we define the constant term of $K(t,x,y)$ along P as follows,

$$K_P(t,x,y)={\rm vol}(\Gamma_N\backslash N)^{-1}\int_{\Gamma_N\backslash N}\sum_{\gamma\in Z\backslash\Gamma_P}\hat f_{t,r}(x^{-1}\gamma ny)\,\chi(\gamma)\,{\rm pr}^P\,dn.$$

For $u \in \mathbb{R}$, let $\alpha_P(u)$ be the characteristic function of $\{x \in G \,|\, H_P(x) + u_P > u\}$, which projects on certain region $\mathcal{C}_{P,u} \subset \Gamma\backslash G$ for a large u. Then the truncation of $K(x,x)$ is defined by

$$\Lambda_u K(t,x,x) := K(t,x,x) - \sum_{P \in \mathfrak{P}} \alpha_P(u) K_P(t,x,x),$$

which is an automorphic form over $\Gamma\backslash G$.

Proposition 4.1. (Maass-Selberg Relation) *For $u \gg 0$, we have*

$$\begin{aligned}(4.4)\quad &\int_{\Gamma\backslash G} \mathrm{tr}\big(\Lambda_u K(t,x,x)\big)\, dx \\ &= u\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\big(\pi^{\mathrm{cst}}_{\frac{1}{2}+i\tau}(f_{t,r})\big)\, d\tau + \mathrm{Tr}\big(\pi^{\mathrm{dis}}(f_{t,r})\big) + \frac{1}{4}\mathrm{Tr}\big(C(\tfrac{1}{2})\pi^{\mathrm{cst}}_{\frac{1}{2}}(f_{t,r})\big) \\ &\qquad - \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\big(C'(\tfrac{1}{2}-i\tau)C(\tfrac{1}{2}+i\tau)\pi^{\mathrm{cst}}_{\frac{1}{2}+i\tau}(f_{t,r})\big)\, d\tau.\end{aligned}$$

Proof. For a test function with compact support and K-finite condition, we can prove this proposition just following the argument in p.58–60 of [14]. Then this can be generalized easily to our test function $f_{t,r}$ as in proof of the theorem 25 of [14]. The finiteness of the integrand of the integrals on the right hand side follows from Theorem 1.1. □

From Proposition 4.1, one can see that the first term on the right side of (4.4) is blowing up as $u \to \infty$. Hence it is natural to remove this term in the following definition,

$$\begin{aligned}(4.5)\quad \mathrm{Tr}\big(D_r e^{-tD_r^2}\big) :=& \mathrm{Tr}\big(\pi^{\mathrm{dis}}(f_{t,r})\big) + \frac{1}{4}\mathrm{Tr}\big(C(\tfrac{1}{2})\pi^{\mathrm{cst}}_{\frac{1}{2}}(f_{t,r})\big) \\ &- \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\big(C'(\tfrac{1}{2}-i\tau)C(\tfrac{1}{2}+i\tau)\pi^{\mathrm{cst}}_{\frac{1}{2}+i\tau}(f_{t,r})\big)\, d\tau.\end{aligned}$$

This regularized trace is the essentially same as the b-trace of Melrose [20], and is related with the geometric side of the Selberg trace formula as we will see in Proposition 4.2. Denote

$$h(\tau) = \Theta_{\frac{1}{2}+i\tau}(f), \quad h(n) = \Theta_n(f)$$

where $\Theta_{\frac{1}{2}+i\tau}(f) := \mathrm{Tr}\big(\pi_{\frac{1}{2}+i\tau}(f)\big)$ for a principal series representation $\pi_{\frac{1}{2}+i\tau}$, and $\Theta_n(f) := \mathrm{Tr}\big(\pi_n(f)\big)$ for a discrete series representation π_n. An operator $J(s)$ over $\mathsf{H} = L^2(Z\backslash K)$ is defined by

$$(4.6)\qquad J(s)\phi_m = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(s)\Gamma(s-\frac{1}{2})}{\Gamma(s+\frac{m}{2})\Gamma(s-\frac{m}{2})}\,\phi_m$$

for the basis $\phi_m(k_\theta) = e^{im\theta} \in \mathsf{H}$ where $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. The following proposition follows from Theorem 13 and Lemma 24 in [14].

Proposition 4.2. (Selberg Trace Formula)

$$\begin{aligned}\operatorname{Tr}\big(D_r e^{-tD_r^2}\big) =& \frac{\operatorname{vol}(\Gamma\backslash G)}{2\pi}\Big(\int_{-\infty}^{\infty} \tau\tanh(\pi\tau)\, h_{t,r}(\tau)\, d\tau + \sum_{n\equiv 0(\mathrm{mod} 2)} (|n|-1) h_{t,r}(n)\Big) \quad (4.7)\\ &+ \sum_{[\gamma]\in Z\backslash\Gamma_{\mathrm{hyp}}} \frac{\operatorname{tr}\big(\chi(\gamma)\big)\, u_\gamma}{4\pi[\Gamma_\gamma : Z]\sinh\frac{u_\gamma}{2}} \int_{-\infty}^{\infty} \cos(u_\gamma\tau)\, h_{t,r}(\tau)\, d\tau\\ &- 2\kappa^{\mathrm{t}}\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(1+2i\tau) h_{t,r}(\tau)\, d\tau + \frac{1}{2}\sum_{n\equiv 0(\mathrm{mod} 2)} h_{t,r}(n)\Big)\\ &+ 2(\kappa-\kappa^{\mathrm{t}})\frac{\log 2}{2\pi}\int_{-\infty}^{\infty} h_{t,r}(\tau)\, d\tau \quad (4.8)\\ &+ \frac{\kappa^{\mathrm{t}}}{2} h_{t,r}(0) - \frac{1}{4\pi}\mathrm{p.v.}\int_{-\infty}^{\infty} \operatorname{Tr}\big(J(\tfrac{1}{2}+i\tau)^{-1}J'(\tfrac{1}{2}+i\tau)\pi_{\frac{1}{2}+i\tau}(f_{t,r})\big)\, d\tau\end{aligned}$$

where $h_{t,r}(\tau), h_{t,r}(n)$ are defined for $f_{t,r}$, the sum $\sum_{[\gamma]\in\Gamma_{\mathrm{hyp}}}$ is given over the Γ-conjugacy class of hyperbolic elements γ conjugate to a_{u_γ}, and $\psi(z) = \Gamma'(z)\Gamma(z)^{-1}$.

5. Fourier transforms $h_{t,r}(\tau), h_{t,r}(n)$

In this section, we compute $h_{t,r}(\tau), h_{t,r}(n)$ which are needed to analyze the right hand side of the Selberg trace formula.

First, let us consider $h_{t,r}(\tau)$. For this, recall

$$h_{t,r}(\tau) = \operatorname{Tr}\big(\pi_{\frac{1}{2}+i\tau}(f_{t,r})\big) = \sum_{n=1}^{\infty}\int_G f_{t,r}(g)\left(\pi_{\frac{1}{2}+i\tau}(g)\xi_n, \xi_n\right) dg \tag{5.1}$$

where $\{\xi_n\}_{n=1}^{\infty}$ is the orthonormal basis of the representation space of $\pi_{\frac{1}{2}+i\tau}$, which is given by the union of the following spaces indexed by $m\in\mathbb{Z}$ for $s=\frac{1}{2}+i\tau$,

$$\mathsf{H}(s,m) := \left\{\, \phi_s\in\mathsf{H}_s \,\middle|\, \phi_s(na_uk_\theta) = e^{su}e^{im\theta} \ \text{ for } na_uk_\theta\in N_0A_0K \,\right\}.$$

Since $\widetilde{D}_r$ is Z-invariant, the Fourier transform $h_{t,r}(\tau)$ is nontrivial only if m is an even number. Recalling

$$\widetilde{D}_r = \Big(\frac{2-r^2}{2r}\Big) + \begin{pmatrix} r^{-1}Z & 2X_- \\ -2X_+ & -r^{-1}Z\end{pmatrix}$$

the problem is again reduced to the following lemma, which can be obtained applying the equalities in (3.3).

Lemma 5.1. *We have*

$$Zf = mf, \quad X_\pm f = -\frac{i}{2}\big(m\pm 2s\big)e^{\pm 2i\theta} f \qquad \textit{for } f\in\mathsf{H}(s,m).$$

The second equality in Lemma 5.1 implies that $X_\pm$ maps $\mathsf{H}(s,m)$ to $\mathsf{H}(s,m\pm 2)$. From these facts,

$$\widetilde{D}_r\begin{pmatrix}\phi_{\tau,m-2}\\ \phi_{\tau,m}\end{pmatrix} = \begin{pmatrix} r^{-1}(m-2)+2^{-1}\ell & -i(m-1-2i\tau)\\ i(m-1+2i\tau) & -r^{-1}m+2^{-1}\ell\end{pmatrix}\begin{pmatrix}\phi_{\tau,m-2}\\ \phi_{\tau,m}\end{pmatrix}$$

where $\ell = \frac{2-r^2}{r}$ and $\phi_{\tau,m-2} \in \mathsf{H}(\frac{1}{2}+i\tau, m-2), \phi_{\tau,m} \in \mathsf{H}(\frac{1}{2}+i\tau, m)$. Hence the action of $\widetilde{D}_r$ on $\mathsf{H}(\frac{1}{2}+i\tau, m-2) \oplus \mathsf{H}(\frac{1}{2}+i\tau, m)$ is given by the roots of

$$\lambda^2 + r\lambda + \frac{r^2}{4} - \frac{(m-1)^2}{r^2} = (m-1)^2 + 4\tau^2, \tag{5.2}$$

that is,

$$\lambda_{\pm}(\tau, m) = -\frac{r}{2} \pm \big((m-1)^2(1+r^{-2}) + 4\tau^2\big)^{1/2} \qquad \text{for } m \in 2\,\mathbb{Z}, \tau \in \mathbb{R}^+.$$

Therefore we have

Lemma 5.2.

$$h_{t,r}(\tau) = \Theta_{\frac{1}{2}+i\tau}(f_{t,r}) = \sum_{m\in 2\mathbb{Z}} \big(\lambda_+(\tau,m)e^{-t\lambda_+(\tau,m)^2} + \lambda_-(\tau,m)e^{-t\lambda_-(\tau,m)^2}\big).$$

Just repeating the above computation applied to the Eisenstein series $E(P, \phi_m, s)$, we can also prove Theorem 1.1.

Next we compute $h_{t,r}(n)$ for the discrete series representation π_n. We review the discrete series representations of $G = \mathrm{SL}(2,\mathbb{R})$. For this it is more convenient to use the Lie group $\mathrm{SU}(1,1)$ which is conjugate to $\mathrm{SL}(2,\mathbb{R})$ within $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \mathrm{SU}(1,1) \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} = \mathrm{SL}(2,\mathbb{R}).$$

Here

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \,\middle|\, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Then the holomorphic discrete series π_n ($n \in \mathbb{N}$) as a representation of $\mathrm{SU}(1,1)$ acts on analytic functions on the disc by

$$\pi_n \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} f(z) = (-\beta z + \overline{\alpha})^{-n} f\left(\frac{\alpha z - \overline{\beta}}{-\beta z + \overline{\alpha}}\right),$$

and the norm, except for a constant factor, is given by

$$||f|| = \begin{cases} \int_{|z|<1} |f(z)|^2 \,(1-|z|^2)^{n-2}\, dz & \text{for } n \geq 2 \\ \sup_{0\leq r<1} \int_0^{2\pi} \left|f(re^{i\theta})\right|^2 \, d\theta & \text{for } n = 1 \end{cases}.$$

The anti-holomorphic discrete series π_n ($n \in -\mathbb{N}$) as a representation of $\mathrm{SU}(1,1)$ acts on analytic functions on the disc by

$$\pi_n \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} f(z) = (-\overline{\beta} z + \alpha)^{-n} f\left(\frac{\overline{\alpha} z - \beta}{-\overline{\beta} z + \alpha}\right)$$

with the same norm.

Lemma 5.3. *For the basis $\{z^N\}_{N\in\{0\}\cup\mathbb{N}}$ of the space of analytic functions on the disc, we have*

$$X_+ z^N = (N+n) z^{N+1}, \quad X_- z^N = -N z^{N-1},$$
$$Z\, z^N = (2N+n)\, z^N \qquad \text{by the action of } \pi_n,\ n \in \mathbb{N},$$

$$X_+ z^N = -N z^{N+1}, \quad X_- z^N = (N+n) z^{N-1},$$
$$Z\, z^N = -(2N+n) z^N \qquad \textit{by the action of } \pi_n,\ n \in -\mathbb{N}.$$

Proof. By elementary computations, we can see that the subgroups generating K, H, A are transformed as follows:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \longrightarrow \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$
$$\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh t & i\sinh t \\ -i\sinh t & \cosh t \end{pmatrix}$$
$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

where the matrices on the right side denote elements in $\mathrm{SU}(1,1)$. To see the action of K under π_n for $n \in \mathbb{N}$, let us consider

$$\pi_n \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} z^N = e^{(2N+n)i\theta} z^N, \tag{5.3}$$

which implies

$$Z\, z^N = -i \frac{d}{d\theta}\Big|_{\theta=0} \pi_n \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} z^N = (2N+n)\, z^N.$$

In a similar way, we can show that the action π_n for $n \in \mathbb{N}$ by H, A are given by

$$H\, z^N = i(N+n) z^{N+1} + iN z^{N-1}, \qquad A\, z^N = (N+n) z^{N+1} - N z^{N-1}.$$

These imply the equalities for π_n for $n \in \mathbb{N}$. The case for π_n for $n \in -\mathbb{N}$ can be obtained by taking the complex conjugates of equalities for the action of π_n for $n \in \mathbb{N}$. $\square$

Now we consider the action of $\widetilde{D}_r$ under π_n for $n \in \mathbb{N}$. From (5.3), we can see that z^N are the K-type vectors of weight m if $m = 2N + n$. By Proposition 5.3, over (α, β) for K-type $(m-2), m$ vectors α, β in the representation space of π_n the Dirac operator $\widetilde{D}_r$ acts by

$$\left(\frac{2-r^2}{2r}\right)\mathrm{Id} + \begin{pmatrix} r^{-1}(m-2) & n-m \\ -(n+m-2) & -r^{-1}m \end{pmatrix}$$

noting $N = (m-n)/2$. We have two cases: First, if β is not the minimal K-type for π_n, that is, $m \gneqq n$, as in the derivation of (5.2), we can obtain the corresponding eigenvalue equation

$$\lambda^2 + r\lambda + \frac{r^2}{4} - \frac{(m-1)^2}{r^2} = (m-1)^2 - (n-1)^2.$$

Hence

$$\lambda_\pm(n,m) = -\frac{r}{2} \pm \big((m-1)^2(1+r^{-2}) - (n-1)^2\big)^{1/2} \qquad \text{for } m = n+2, n+4, \ldots.$$

Second, if β is the minimal K-type for π_n, that is, K-type $m = n$ vector, then α is just trivial. Hence, the eigenvalue is given by

$$\lambda(n) = -\frac{r}{2} + \frac{1-n}{r}.$$

Repeating the same procedure as in the case of π_n for $n \in \mathbb{N}$, we can obtain the same eigenvalues $\lambda_\pm(n,m)$, $\lambda(n)$ for π_n for $n \in -\mathbb{N}$. In our case, n should be an even number since Γ contains $-\mathrm{Id}$. Combining all the facts derived in the above, we have

Lemma 5.4. *For $n \in 2\mathbb{N}, h(n) = \Theta_n(f)$, we have*

$$h_{t,r}(n) = h_{t,r}(-n) = \Big(\lambda(n)e^{-t\lambda(n)^2} + \sum_{m\in n+2\mathbb{N}} \big(\lambda_+(n,m)e^{-t\lambda_+(n,m)^2} + \lambda_-(n,m)e^{-t\lambda_-(n,m)^2}\big)\Big)$$

where $\lambda(n) = -\frac{r}{2} + \frac{1-n}{r}$, $\lambda_\pm(n,m) = -\frac{r}{2} \pm \big((m-1)^2(1+r^{-2}) - (n-1)^2\big)^{1/2}$.

6. Eta function of D_r: Principal series part

Now we study the eta function defined by

$$\eta(D_r, s) := \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\mathrm{Tr}\big(D_r e^{-tD_r^2}\big)\,dt$$

for $\Re(s) \gg 0$ and r near 0. First let us recall that the bottom of each branch of continuous spectrum of D_r goes to ∞ as $r \to 0$ by Theorem 1.1. Hence for a small $r > 0$, $\mathrm{Tr}\big(D_r e^{-tD_r^2}\big)$ decays exponentially as $t \to \infty$. To analyze $\mathrm{Tr}\big(D_r e^{-tD_r^2}\big)$ in more detail, we apply Proposition 4.2 which relates $\mathrm{Tr}\big(D_r e^{-tD_r^2}\big)$ with the geometric side. Note that this geometric side can be decomposed into two parts:

$$\begin{aligned}\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big) =& \mathrm{Tr}\big(D_r e^{-tD_r^2}\big) - \mathrm{Tr}_{\mathrm{d}}\big(D_r e^{-tD_r^2}\big),\\ \mathrm{Tr}_{\mathrm{d}}\big(D_r e^{-tD_r^2}\big) =& \frac{\mathrm{vol}(\Gamma\backslash G)}{2\pi}\sum_{n\equiv 0(\mathrm{mod}2)}(|n|-1)h_{t,r}(n) - \kappa^{\mathrm{t}}\sum_{n\equiv 0(\mathrm{mod}2)} h_{t,r}(n)\end{aligned}$$

and accordingly we also decompose the eta function $\eta(D_r, s)$ into

$$\eta(D_r, s) = \eta_{\mathrm{p}}(D_r, s) + \eta_{\mathrm{d}}(D_r, s).$$

The principal part of the eta function $\eta_{\mathrm{p}}(D_r, s)$ is studied in this section and the other part $\eta_{\mathrm{d}}(D_r, s)$ will be considered in the next section.

We start with the following lemma.

Lemma 6.1. *Putting* $I(m,r,\tau) = \big((m-1)^2(1+r^{-2}) + 4\tau^2\big)^{\frac{1}{2}}$,

$$h_{t,r}(\tau) = \exp\big(-\frac{r^2}{4}t - 4\tau^2 t\big) \cdot \sum_{m\in 2\mathbb{Z}} e^{-(m-1)^2(1+r^{-2})t}\Big(\sum_{k=0}^{\infty}\big(-r\frac{(rt)^{2k}}{(2k)!} + 2\frac{(rt)^{2k-1}}{(2k-1)!}\big)\, I(m,r,\tau)^{2k}\Big)$$

where the term $\frac{(rt)^{2k-1}}{(2k-1)!}$ for $k = 0$ vanishes and for $t \in [0,1]$ and $r \in (0,1]$ the following estimate holds,

$$|h_{t,r}(\tau)| \le 2r\exp\big(-\frac{r^2}{4}t - 4\tau^2 t\big) \cdot \sum_{m\in 2\mathbb{Z}} e^{-(m-1)^2(1+r^{-2})t}\Big(1 + I(m,r,\tau)^2 + e^{I(m,r,\tau)^2 rt}\Big). \tag{6.1}$$

Proof. We can rewrite $h_{t,r}(\tau)$ as follows,

$$\begin{aligned}(6.2)\quad h_{t,r}(\tau) = \exp\big(-\frac{r^2}{4}t-4\tau^2 t\big)\sum_{m\in 2\mathbb{Z}} e^{-(m-1)^2(1+r^{-2})t} \\ \cdot\Big(-\frac{r}{2}\big(e^{I(m,r,\tau)rt}+e^{-I(m,r,\tau)rt}\big)+I(m,r,\tau)\big(e^{I(m,r,\tau)rt}-e^{-I(m,r,\tau)rt}\big)\Big).\end{aligned}$$

Now the Taylor expansion of $e^{I(m,r,\tau)rt} \pm e^{-I(m,r,\tau)rt}$ gives us the claimed form of the first equality. To prove the second estimate, we note that

$$\begin{aligned}&\sum_{k=0}^{\infty}\big(-r\frac{(rt)^{2k}}{(2k)!}+2\frac{(rt)^{2k-1}}{(2k-1)!}\big)\, I(m,r,\tau)^{2k} \\ &\qquad = -r+rt\big(2-\frac{r^2 t}{2}\big)I(m,r,\tau)^2+\sum_{k=2}^{\infty}\frac{(rt)^{2k-1}}{(2k-1)!}\big(2-\frac{r^2 t}{2k}\big)I(m,r,\tau)^{2k}.\end{aligned}$$

For $t\in[0,1]$ and $r\in(0,1]$, observe that

$$\sum_{k=2}^{\infty}\frac{(rt)^{2k-1}}{(2k-1)!}\big(2-\frac{r^2t}{2k}\big)I(m,r,\tau)^{2k} \;\le\; 2r\sum_{k=0}^{\infty}\frac{(rt)^k}{k!}I(m,r,\tau)^{2k},$$

from which it is easy to derive the estimate. □

Our first task in this section is to get the asymptotic expansion of $\mathrm{Tr_p}\big(D_r e^{-tD_r^2}\big)$ as $t\to 0$. By Lemma 6.1, we can rewrite the first part of $\mathrm{Tr_p}\big(D_r e^{-tD_r^2}\big)$ as follows,

$$\begin{aligned}(6.3)\qquad \int_{-\infty}^{\infty}\tau\tanh(\pi\tau)\, h_{t,r}(\tau)\, d\tau = &\sum_{m\in 2\mathbb{Z}}\exp\big(-\frac{r^2}{4}t-(m-1)^2(1+r^{-2})t\big)\\ &\cdot\sum_{k=0}^{\infty}\sum_{k=p+q}\big(a_{k,p,q}(r)t^{2k}+b_{k,p,q}(r)t^{2k-1}\big)\\ &\cdot(m-1)^{2p}(1+r^{-2})^p\int_{-\infty}^{\infty}\tau\tanh(\pi\tau)\,(2\tau)^{2q}e^{-4\tau^2 t}\,d\tau\end{aligned}$$

where $a_{k,p,q}(r), b_{k,p,q}(r)$ ($b_{0,p,q}(r)=0$) depend only on r and are of order $\mathrm{O}(r)$ for small $r>0$. The integral in the last line can be handled as follows,

$$\begin{aligned}\int_{-\infty}^{\infty}\tau\tanh(\pi\tau)\,(2\tau)^{2q}e^{-4\tau^2t}\,d\tau =&(-1)^q\partial_t^q\int_{-\infty}^{\infty}\tau\tanh(\pi\tau)\,e^{-4\tau^2t}\,d\tau\\ =&(-1)^q\partial_t^q\int_0^{\infty}\tanh(\pi\sqrt{x})e^{-4tx}\,dx\\ =&(-1)^q\partial_t^q\frac{\pi}{8t}\int_0^{\infty}\Big(\sum_{k=0}^{\infty}\frac{(-4tx)^k}{k!}\Big)\frac{\cosh^{-2}(\pi\sqrt{x})}{\sqrt{x}}\,dx.\end{aligned}$$

Hence we have

$$(6.4)\qquad \int_{-\infty}^{\infty}\tau\tanh(\pi\tau)\,(2\tau)^{2q}e^{-4\tau^2t}\,d\tau \sim \sum_{k=0}^{\infty}a_k\, t^{-q-1+k}\qquad \text{as}\quad t\to 0$$

where a_k are independent of r. Now we note the following equalities for the first and the third factors on the right hand side of (6.3),

$$\begin{aligned}
&\sum_{m\in 2\mathbb{Z}} (m-1)^{2p}(1+r^{-2})^p \exp\big(-(m-1)^2(1+r^{-2})t\big) \qquad (6.5)\\
=&(-1)^p\partial_t^p \sum_{m\in\mathbb{Z}} \exp\big(-4(m-\frac{1}{2})^2(1+r^{-2})t\big)\\
=&(-1)^p\partial_t^p \sum_{m\in\mathbb{Z}} (-1)^m \frac{\sqrt{\pi}}{2\sqrt{(1+r^{-2})t}} \exp\big(-\frac{\pi^2m^2}{4(1+r^{-2})t}\big)\\
=&(-1)^p\partial_t^p \left(\frac{\sqrt{\pi}r}{2\sqrt{(1+r^2)t}} + \sum_{m\in\mathbb{Z}-\{0\}} (-1)^m \frac{\sqrt{\pi}}{2\sqrt{(1+r^{-2})t}} \exp\big(-\frac{\pi^2m^2}{4(1+r^{-2})t}\big)\right)
\end{aligned}$$

where the second equality is the Poisson summation formula. Note that the terms for nonzero m in the last line of (6.5) decays exponentially as $t\to 0$, so that small time asymptotics is given by the first term in the last line of (6.5). Therefore we have

$$\sum_{m\in 2\mathbb{Z}} (m-1)^{2p}(1+r^{-2})^p \exp\big(-(m-1)^2(1+r^{-2})t\big) \sim a(r)\, t^{-\frac{1}{2}-p} \qquad \text{as} \quad t\to 0 \tag{6.6}$$

where $a(r)$ depends only on r and $\mathrm{O}(r)$ for small $r>0$.

By (6.3) and the asymptotic expansions in (6.4), (6.6), taking care of r-dependence of their coefficients, we can conclude

$$\int_{-\infty}^{\infty} \tau \tanh(\pi\tau)\, h_{t,r}(\tau)\, d\tau \sim \sum_{k=0}^{\infty} a_k(r)\, t^{-\frac{3}{2}+k} \qquad \text{as} \quad t\to 0 \tag{6.7}$$

where $a_k(r)$ depends only on r and is of $\mathrm{O}(r^2)$ for small $r>0$.

Now for the second part of $\mathrm{Tr_p}\big(D_r e^{-tD_r^2}\big)$, we repeat the above process and noting

$$\begin{aligned}
\int_{-\infty}^{\infty} \cos(u_\gamma\tau)\,(2\tau)^{2q} e^{-4\tau^2 t}\, d\tau &= (-1)^q\partial_t^q \int_{-\infty}^{\infty} \cos(u_\gamma\tau)\, e^{-4\tau^2 t}\, d\tau\\
&= (-1)^q\partial_t^q \Big(\frac{\sqrt{\pi}}{\sqrt{t}} \exp(-\frac{u_\gamma^2}{4t})\Big),
\end{aligned}$$

we can see that this term does not contribute to the asymptotics as $t\to 0$.

To deal with the third part of $\mathrm{Tr_p}\big(D_r e^{-tD_r^2}\big)$, we recall

$$\psi(1+z) \sim \log z + \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} z^{-2k} \qquad \text{as} \quad z\to\infty \tag{6.8}$$

where B_{2k} is the Bernoulli's number, which implies

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi(1+2i\tau)\,(2\tau)^{2q} e^{-4\tau^2 t}\, d\tau \qquad (6.9)\\
&\qquad \sim b\, t^{-\frac{1}{2}-q} \log t + c + \sum_{k=0}^{\infty} a_k\, t^{-q-\frac{1}{2}+k} \qquad \text{as} \quad t\to 0
\end{aligned}$$

where the constant c vanishes unless $q = 0$. Proceeding as before and using (6.9),

$$\int_{-\infty}^{\infty} \psi(1+2i\tau) h_{t,r}(\tau)\, d\tau \sim \sum_{k=0}^{\infty} a_k(r)\, t^{-1+\frac{k}{2}} + b_k(r)\, t^{-1+k} \log t \quad \text{as} \quad t \to 0 \tag{6.10}$$

where $a_k(r), b_k(r)$ depend only on r and is of $\mathrm{O}(r^2)$ for small $r > 0$.

For the fourth part of $\mathrm{Tr_p}\big(D_r e^{-tD_r^2}\big)$, it is also easy to get the following asymptotic expansion

$$\int_{-\infty}^{\infty} h_{t,r}(\tau)\, d\tau \sim \sum_{k=0}^{\infty} a_k(r)\, t^{-1+k} \qquad \text{as} \quad t \to 0 \tag{6.11}$$

where $a_k(r)$ depends only on r and is of $\mathrm{O}(r^2)$ for small $r > 0$.

Now it is easy to see that the next term $\frac{\kappa_t}{2} h_{t,r}(0)$ contributes to the small time asymptotics by (6.7) with the first term $a_1(r) t^{-\frac{1}{2}}$.

By (4.6), the integrand of the last integral of the geometric side can be expressed by

$$\psi\big(\frac{1}{2} + i\tau\big) + \psi\big(i\tau\big) - \psi\big(\frac{1+m}{2} + i\tau\big) - \psi\big(\frac{1-m}{2} + i\tau\big). \tag{6.12}$$

Using the following formulas about $\psi(z)$,

$$\psi(z+1) = \frac{1}{z} + \psi(z), \qquad \psi(z) + \psi(z + \frac{1}{2}) = 2\big(\psi(2z) - \log 2\big), \tag{6.13}$$

the terms in (6.12) can be rewritten as

$$\begin{aligned} &2\big(\psi(1+i\tau) - \psi(1+2i\tau)\big) - \frac{1}{i\tau} + 2\log 2 \\ &\qquad\qquad - 4\Big(\frac{1}{1+4\tau^2} + \frac{3}{3^2+4\tau^2} + \ldots + \frac{m-1}{(m-1)^2+4\tau^2}\Big). \end{aligned}$$

The terms in the first line gives us the asymptotics as (6.10). The terms in the second line also can be handled as in a similar way and we can show that these term gives us the asymptotics

$$\sum_{k=0}^{\infty} a_k(r)\, t^{-\frac{1+k}{2}} \qquad \text{as} \quad t \to 0 \tag{6.14}$$

where $a_k(r)$ depends only on r and is of $\mathrm{O}(r^2)$ for small $r > 0$. Combining (6.7), (6.10), (6.14) and facts derived in the above, we obtain

Theorem 6.2. *The small time asymptotics is given by*

$$\mathrm{Tr_p}\big(D_r e^{-tD_r^2}\big) \sim \sum_{k=0}^{\infty} a_k(r)\, t^{\frac{-3+k}{2}} + b_k(r)\, t^{-1+k} \log t \quad \textit{as} \quad t \to 0 \tag{6.15}$$

where $a_k(r), b_k(r)$ depend only on r and is of $\mathrm{O}(r^2)$ for small $r > 0$. In particular, if $\kappa^t = 0$, it has the following simple form,

$$\mathrm{Tr_p}\big(D_r e^{-tD_r^2}\big) \sim \sum_{k=0}^{\infty} a_k(r)\, t^{-\frac{3+k}{2}} \qquad \textit{as} \quad t \to 0.$$

This theorem also immediately implies

Theorem 6.3. *For a sufficiently small $r > 0$, the function $\eta_{\mathrm{p}}(D_r, s)$ defined for $\Re(s) \gg 0$ has the meromorphic extension over $\mathbb{C}$ and may have a double pole at $s = 1$ and simple poles at $-\mathbb{N} \cup \{0, 2\}$. In particular, if $\kappa^{\mathrm{t}} = 0$, $\eta_{\mathrm{p}}(D_r, s)$ may have only the simple poles at $-\mathbb{N} \cup \{0, 1, 2\}$.*

In the view of Theorem 6.3, it is natural to define the principal part of the eta invariant of D_r by

$$\eta_{\mathrm{p}}(D_r) := \Big(\eta_{\mathrm{p}}(D_r, s) - \frac{r_0}{s}\Big)\Big|_{s=0}$$

where r_0 is the residue of the simple pole of $\eta_{\mathrm{p}}(D_r, s)$ at $s = 0$. Now let us consider the adiabatic limit of $\eta_{\mathrm{p}}(D_r)$ as $r \to 0$. For this, we need

Proposition 6.4. *As $r \to 0$, $\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)$ converges to 0 for $t \in (0, \infty)$, and $t^{\frac{3}{2}}\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)$ converges to 0 uniformly for $t \in [0, 1]$.*

Proof. From the expression of $h_{t,r}(\tau)$ in (6.2), we can see

$$|h_{t,r}(\tau)| \,\le\, C\, \exp\big(-\frac{r^2}{4}t - 4\tau^2 t\big) \sum_{m \in 2\mathbb{Z}} e^{-c(m-1)^2(1+r^{-2})t} \qquad \text{for small } r > 0$$

where C, c are the positive constants that do not depend on r, τ. Hence, the integral

$$\int_{-\infty}^{\infty} \tau \tanh(\pi\tau)\, h_{t,r}(\tau)\, d\tau$$

vanishes as $r \to 0$ by the dominated convergence theorem. The same argument holds for other terms defining $\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)$. Hence $\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)$ converges to 0 as $r \to 0$. The uniform convergence of $t^{\frac{3}{2}}\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)$ follows from the following estimate

$$\Big| t^{\frac{3}{2}}\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)\Big| \le Cr^2 \qquad \text{for } \ t \in [0, 1], \tag{6.16}$$

which also follows easily from (6.1) and (6.5). □

Now we have

Theorem 6.5.

$$\lim_{r \to 0} \eta_{\mathrm{p}}(D_r) = 0.$$

Proof. Let us consider

$$\begin{aligned}\eta_{\mathrm{p}}(D_r) =& \frac{1}{\sqrt{\pi}} \int_1^{\infty} t^{-\frac{1}{2}}\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)\, dt \\ &+ \Big(\frac{1}{\Gamma(\frac{s+1}{2})} \int_0^1 t^{\frac{s-1}{2}}\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)\, dt - \frac{r_0}{s}\Big)\Big|_{s=0}.\end{aligned}$$

For the integration over $[1, \infty)$, recalling that $\mathrm{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)$ is exponentially decaying as $t \to \infty$, it is easy to see that this part vanishes as $r \to 0$ by Proposition 6.4 and the dominated convergence theorem. By (6.15) the meromorphic extension of the integral $\int_0^1 \cdot\, dt$ has the

following form for $\Re(s) \geq -\epsilon$ with small $\epsilon > 0$,

$$
\begin{aligned}
(6.17) \qquad \int_0^1 t^{\frac{s-1}{2}} \operatorname{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)\, dt \;=\; & \frac{2a_0}{s-2} + \frac{2a_1}{s-1} - \frac{4b_0}{(s-1)^2} + \frac{2a_2}{s} \\
& + \frac{2a_3}{s+1} - \frac{4b_1}{(s+1)^2} + \int_0^1 t^{\frac{s-1}{2}} \operatorname{Tr}^*_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big)\, dt,
\end{aligned}
$$

where

$$
\operatorname{Tr}^*_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big) := \operatorname{Tr}_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big) - a_0\, t^{-\frac{3}{2}} - a_1\, t^{-1} - b_0 t^{-1}\log t - a_2\, t^{-\frac{1}{2}} - a_3 - b_1 \log t.
$$

By Theorem 6.2, all the coefficients $a_0, a_1, a_2, a_3, b_0, b_1$ (as function of variable r) vanish as $r \to 0$. Hence putting $s = 0$ except the term $\frac{2a_2}{s}$, we can see that $-a_0 - 2a_1 - 4b_1 + 2a_3 - 4b_1$ vanishes as $r \to 0$. For the last integral with $s = 0$ also vanishes as $r \to 0$ since

$$
\Big| t^{-\frac{1}{2}} \operatorname{Tr}^*_{\mathrm{p}}\big(D_r e^{-tD_r^2}\big) \Big| \leq C r^2 \qquad \text{for} \quad t \in [0,1],
$$

which follows from (6.16) and the coefficients $a_0, a_1, a_2, a_3, b_0, b_1$ vanish as order of r^2. This completes the proof.

□

7. Eta function of D_r: Discrete series part

In this section we study the discrete part of the eta function $\eta_{\mathrm{d}}(D_r, s)$ when $r > 0$ is sufficiently small.

First, from Lemma 5.4, let us recall that $h_{t,r}(n)$ is given by $\lambda(n)$'s and $\lambda_{\pm}(n,m)$'s and we decompose $\operatorname{Tr}_{\mathrm{d}}\big(D_r e^{-tD_r^2}\big)$ into the corresponding two parts. Then we also have

$$
\eta_{\mathrm{d}}(D_r, s) = \eta^1_{\mathrm{d}}(D_r, s) + \eta^2_{\mathrm{d}}(D_r, s) \qquad \text{for} \quad \Re(s) \gg 0
$$

where

$$
\eta^1_{\mathrm{d}}(D_r, s) = r^s\,(2g - 2 + \kappa)\Big(- \sum_{k=1}^{\infty} \frac{2(2k-1)}{(2k - 1 + \frac{r^2}{2})^s} \Big) + r^s \kappa^{\mathrm{t}} \Big(\sum_{k=1}^{\infty} \frac{2}{(2k - 1 + \frac{r^2}{2})^s} \Big)
$$

$$
\begin{aligned}
\eta^2_{d}(D_r, s) = (2g - 2 + \kappa) \Big(2 \sum_{k=1}^{\infty} (2k-1) \sum_{\ell \in k + \mathbb{N}} \lambda_+(2k, 2\ell)^{-s} - \lambda_-(2k, 2\ell)^{-s} \Big) \\
- \kappa^{\mathrm{t}} \Big(2 \sum_{k=1}^{\infty} \sum_{\ell \in k + \mathbb{N}} \lambda_+(2k, 2\ell)^{-s} - \lambda_-(2k, 2\ell)^{-s} \Big).
\end{aligned}
$$

Here we used the fact

$$
\operatorname{vol}(\Gamma \backslash G) = 2\pi\, (2g - 2 + \kappa)
$$

where the volume of $\Gamma\backslash G$ is given w.r.t. the Haar measure in (4.1) (recall that the volume of the circle K/Z is normalized to be 1).

Now we investigate $\eta^1_{\mathrm{d}}(D_r, s)$. Let us recall the Hurwitz zeta function

$$
\zeta(s, a) = \sum_{k=0}^{\infty} (k + a)^{-s}
$$

which has a meromorphic extension to the whole $\mathbb{C}$ with a simple pole at $s = 1$. If we set

$$\zeta_0(s, a) = \sum_{k=1}^{\infty} (2k - 1 + a)^{-s},$$

then

$$\zeta_0(s, a) = \zeta(s, a) - 2^{-s} \zeta(s, \frac{a}{2}).$$

By these definitions, for $\Re(s) \gg 0$,

$$\eta^1_{\mathrm{d}}(D_r, s) = 2\,(2 - 2g - \kappa)\, r^s \left(\zeta_0(s - 1, \frac{r^2}{2}) - \frac{r^2}{2} \zeta_0(s, \frac{r^2}{2}) \right) + 2\kappa^{\mathrm{t}} r^s \zeta_0(s, \frac{r^2}{2}).$$

The right hand side gives the meromorphic extension of $\eta^1_{\mathrm{d}}(D_r, s)$ over $\mathbb{C}$ with the simple poles at $s = 1, 2$. We can also see that $\eta^1_{\mathrm{d}}(D_r, s)$ is regular at $s = 0$ from this equality. Recalling

$$\zeta(0, a) = \frac{1}{2} - a, \qquad \zeta(-1, a) = -\frac{1}{2} \left(a^2 - a + \frac{1}{6} \right),$$

we can see that

$$\zeta_0(0, a) = -\frac{a}{2}, \qquad \zeta_0(-1, a) = -\frac{1}{4} \left(a^2 - \frac{1}{3} \right).$$

Using these, we obtain

$$\eta^1_{\mathrm{d}}(D_r, 0) = (2 - 2g - \kappa) \left(\frac{1}{6} + \frac{r^4}{8} \right) - \kappa^{\mathrm{t}} \frac{r^2}{2}.$$

Summarizing all these for $\eta^1_{\mathrm{d}}(D_r, s)$, we have

Proposition 7.1. *For a sufficiently small $r > 0$, the function $\eta^1_{\mathrm{d}}(D_r, s)$ define for $\Re(s) \gg 0$ has the meromorphic extension over $\mathbb{C}$ and has the simple poles at $s = 1, 2$. The following equality holds,*

$$\lim_{r \to 0} \eta^1_{\mathrm{d}}(D_r, 0) = \frac{1}{6} (2 - 2g - \kappa). \tag{7.1}$$

To get the meromorphic extension of $\eta^2_{\mathrm{d}}(D_r, s)$ over $\mathbb{C}$, we rewrite this as follows,

$$\eta^2_{\mathrm{d}}(D_r, s) = 2\,(2g - 2 + \kappa)\, r^s f_r(s) - 2\kappa^{\mathrm{t}} r^s g_r(s).$$

Here

$$f_r(s) = \sum_{\ell > k \geq 1} (2k - 1) \Big(\big(q_r(k, \ell) - \frac{r^2}{2}\big)^{-s} - \big(q_r(k, \ell) + \frac{r^2}{2}\big)^{-s} \Big),$$

$$g_r(s) = \sum_{\ell > k \geq 1} \big(q_r(k, \ell) - \frac{r^2}{2}\big)^{-s} - \big(q_r(k, \ell) + \frac{r^2}{2}\big)^{-s}$$

where

$$q_r(k, \ell) = \big((2\ell - 1)^2 (1 + r^2) - r^2 (2k - 1)^2\big)^{\frac{1}{2}}.$$

Now we put $h_r(s) = \sum_{\ell > k \geq 1} (2k - 1) q_r(k, \ell)^{-s}$ which can be written as

$$h_r(s) = \sum_{k \geq 1} (2k - 1)^{1-s} \sum_{\ell > k} (2\ell - 1)^{-s} \Big(\frac{1 + r^2}{(2k - 1)^2} - \frac{r^2}{(2\ell - 1)^2} \Big)^{-\frac{s}{2}}.$$

From this and the above analysis of $\zeta_0(s,0)$, we can see that $h_r(s)$ is holomorphic for $\Re(s)>2$. For the meromorphic extension of $h_r(s)$ over $\mathbb{C}$, we use the identity $a^s=\exp(s\log(1+(a-1)))$ to get

$$\begin{aligned}&\Big(\frac{1+r^2}{(2k-1)^2}-\frac{r^2}{(2\ell-1)^2}\Big)^{-\frac{s}{2}}\\ =&1-\frac{s}{2}\Big(\frac{1+r^2}{(2k-1)^2}-\frac{r^2}{(2\ell-1)^2}-1\Big)+\frac{s}{4}\Big(\frac{1+r^2}{(2k-1)^2}-\frac{r^2}{(2\ell-1)^2}-1\Big)^2+\dots.\end{aligned}$$

From this, we can see that $h_r(s)$ has the meromorphic extension over $\mathbb{C}$ and may have the simple poles at $s=2,1,0,-1,\dots$ with the residues which are continuous w.r.t. r. Using the following equality

$$f_r(s)=\left(r^2sh_r(s+1)+r^6\frac{s(s+1)(s+2)}{24}h_r(s+3)+r^{10}\theta(s,r)\right)$$

where $\theta(s,r)$ is regular at $s=0$ and is continuous at $r=0$, we can conclude that $f_r(s)$ is regular at $s=0$ and the limit of $f_r(0)$ as $r\to 0$ is trivial. In a similar way, we can see that the same conclusion is true for $g_r(s)$. By all these facts, we have

Proposition 7.2. *For a sufficiently small $r>0$, the function $\eta^2_{\mathrm{d}}(D_r,s)$ defined for $\Re(s)\gg 0$ has the meromorphic extension over $\mathbb{C}$ and may have the simple poles at $s=-\mathbb{N}\cup\{1\}$. The following equality holds,*

$$\lim_{r\to 0}\eta^2_{\mathrm{d}}(D_r,0)=0. \tag{7.2}$$

By Proposition 7.1, 7.2, we can define

$$\eta_{\mathrm{d}}(D_r):=\eta_{\mathrm{d}}(D_r,s)\big|_{s=0}=\eta^1_{\mathrm{d}}(D_r,0)+\eta^2_{\mathrm{d}}(D_r,0)$$

and

Theorem 7.3. *For a sufficiently small $r>0$, the discrete part of the eta function $\eta_{\mathrm{d}}(D_r,s)$ has the meromorphic extension over $\mathbb{C}$ and may have the simple poles at $-\mathbb{N}\cup\{1,2\}$. The following equality holds,*

$$\lim_{r\to 0}\eta_{\mathrm{d}}(D_r)=\frac{1}{6}\left(2-2g-\kappa\right).$$

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